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Stability and Growth Estimates for Electric Fields in Non-Conducting Material Dielectrics*

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Employing results derived by the author for solutions of an abstract integro-differential equation in Hilbert space, we obtain stability and growth estimates for electric fields in nonconducting material dielectrics. It is assumed that a linear constitutive equation of Maxwell-Hopkinson type relates the electric field and the electric displacement field in the dielectric; specific results for a simple memory function of exponential type are given.

1. INTRODUCTION

Let (x^i, t), $i = 1, 2, 3$, denote a Lorentz reference frame where (x^i) represent rectangular Cartesian coordinates and t is the time parameter; in this frame of reference the local forms of Maxwell's equations are

$$\frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \mathbf{E} = \mathbf{0}, \qquad \operatorname{div} \mathbf{B} = 0 \tag{1.1a}$$

$$\operatorname{curl} \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t} = \mathbf{0}, \qquad \operatorname{div} \mathbf{D} = 0 \tag{1.1b}$$

provided that the density of free current $\mathbf{J}_F$, and the magnetization $\mathbf{M}$ are each equal to the zero vector and the density of free charge $Q_F = 0$. In (1.1a) and (1.1b) $\mathbf{B}$ is, of course, the magnetic flux density, while $\mathbf{E}$, $\mathbf{H}$, and $\mathbf{D}$ represent the electric field, magnetic intensity, and electric displacement vectors, respectively.

To obtain a determinate system of equations for the fields appearing in (1.1) it is also necessary to append certain constitutive equations, the form of such relations being dependent on the nature of the material in which the electric and magnetic fields occur. For example, in a vacuum we have the classical constitutive relations

$$\mathbf{D} = \epsilon_0 \mathbf{E}, \qquad \mathbf{H} = \mu_0^{-1} \mathbf{B} \tag{1.2}$$

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where ϵ_0, μ_0 are fundamental constants satisfying $\epsilon_0\mu_0 = c^{-2}$, c being the speed of light in a vacuum. The next simplest kind of material in which $(\mathbf{E}, \mathbf{B})$ may occur is a rigid, linear, stationary non-conducting dielectric whose constitutive relations, viz.,

$$\mathbf{D} = \boldsymbol{\epsilon} \cdot \mathbf{E}, \qquad \mathbf{B} = \mathbf{u} \cdot \mathbf{H} \tag{1.3}$$

where given by Maxwell [1] in 1873; in (1.3) $\boldsymbol{\epsilon}$, $\mathbf{u}$ are constant second order tensors which are proportional to the identity tensor if the material is isotropic. As pointed out by Toupin and Rivilin [2] the relations (1.3) do not account for the observed absorbtion and dispersion of electromagnetic waves in non-conductors.

In 1877 Hopkinson [3], in connection with his studies on the residual charge of the Leyden jar (and following a suggestion of Maxwell), proposed a constitutive equation for the electric displacement in a non-conducting dielectric of the form

$$\mathbf{D}(t) = \epsilon\mathbf{E}(t) + \int_{-\infty}^{t} \phi(t-\tau)\,\mathbf{E}(\tau)\,d\tau \tag{1.4}$$

where $\epsilon > 0$ and $\phi(t)$, $t \geqslant 0$, is a decreasing function of t which is continuous for $0 \leqslant t < \infty$. As indicated in [2] Hopkinson was able to correlate his data on the residual charge of Leyden jars by making suitable adjustments of the memory function $\phi(t)$; for instance, he points out in [3] that a suitable memory function for glass would be a linear combination of exponentials with the coefficients in the expansion being dependent upon the silica composition of the material.

We shall be concerned in this paper with the growth behavior of electric fields which occur in non-conducting material dielectrics that are governed by the constitutive hypothesis (1.4); following Davis [4] we append to the system consisting of (1.1) and (1.4) the relation

$$\mathbf{H} = \mu^{-1}\mathbf{B}, \qquad \mu > 0 \tag{1.5}$$

Results concerning continuous dependence of the electric field on perturbations of the memory function ϕ, etc., may be obtained via a suitable interpretation of the abstract results contained in [5].

2. Growth Estimates for an Abstract Integrodifferential Equation in Hilbert Space

Throughout the remainder of this paper we deal with the constitutive relations (1.4), (1.5) and assume, for the sake of convenience, that $\mathbf{E}(t) = \mathbf{0}$ for $t < 0$. Then, as indicated in [4], we may solve (1.4) for $\mathbf{E}(t)$ by the usual of successive approximations and we get

$$\mathbf{E}(t) = \epsilon^{-1}\mathbf{D}(t) + \epsilon^{-1}\int_0^t \Phi(t-\tau)\,\mathbf{D}(\tau)\,d\tau \tag{2.1}$$

where

$$\Phi(t) = \sum_{n=1}^{\infty} (-1)^n \phi^n(t) \tag{2.2}$$

$$\phi^1(t) = \epsilon^{-1}\phi(t) \tag{2.3}$$

$$\phi^n(t) = \int_0^t \phi^1(t-\tau)\,\phi^{n-1}(\tau)\,d\tau, \qquad n \geqslant 2. \tag{2.4}$$

Because of the assumed continuity of $\phi(t)$, $0 \leqslant t < \infty$, $\Phi(t)$ will be in $\mathscr{C}[0, T)$ if the series in (2.2) converges uniformly for $0 \leqslant t < T < \infty$; such uniform convergence will be postulated in the next section where we obtain upper and lower bounds for $\sup_{[0,T)} |\Phi(t)|$ and $\sup_{[0,T)} |\dot{\Phi}(t)|$ in terms of $\sup_{[0,T)} |\phi(t)|$ and $\sup_{[0,T)} |\dot{\phi}(t)|$.

The following simple observation is essential.

LEMMA I (Davis [4]). *In any non-conducting material dielectric for which* (1.4), (1.5) *are valid, and* $\mathbf{E}(t) = \mathbf{0}$, $t < 0$, *the electric field and the electric displacement field satisfy*

$$(\epsilon\mathbf{E} + \phi^*\mathbf{E})_{tt} = \mu^{-1}\Delta\mathbf{E} \tag{2.5}$$

$$\epsilon\mu\mathbf{D}_{tt} = \Delta\mathbf{D} + \Phi^*\Delta\mathbf{D} \tag{2.6}$$

where

$$(\phi^*\mathbf{A})_i(\mathbf{x}, t) = \int_0^t \phi(t-\tau)\,A_i(\mathbf{x}, \tau)\,d\tau \tag{2.7}$$

and for any vector field $\mathbf{A}$

$$\Delta\mathbf{A} = \operatorname{grad}(\operatorname{div}\mathbf{A}) - \operatorname{curl}\operatorname{curl}\mathbf{A}. \tag{2.8}$$

Proof. By virtue of (1.1b), (2.1), (2.2)–(2.4) and the spatial independence of $\phi(t)$,

$$\Delta\mathbf{E} = -\operatorname{curl}\operatorname{curl}\mathbf{E} \tag{2.9}$$

But

$$\operatorname{curl}\mathbf{E} = -\mathbf{B}_t = -\mu\mathbf{H}_t \tag{2.10}$$

by virtue of Maxwell's first equation (1.1a) and our constitutive hypothesis (1.5). Thus

$$\Delta\mathbf{E} = \mu(\operatorname{curl}\mathbf{H}_t) = \mu(\operatorname{curl}\mathbf{H})_t = \mu\mathbf{D}_{tt} \tag{2.11}$$

in view of (1.1b). The integrodifferential equation for $\mathbf{E}(t)$, i.e., (2.5) now follows from direct substitution of (1.4) into (2.11) while (2.6), the integrodifferential equation which governs the evolution of $\mathbf{D}(t)$, follows via direct substitution of (2.1) into (2.11).

Our goal in the present work is to derive stability and growth estimates for solutions $\mathbf{E}(\mathbf{x}, \mathbf{t})$, $\mathbf{D}(\mathbf{x}, \mathbf{t})$ to (2.5) and (2.6) respectively where we assume that

$(\mathbf{x}, t) \in \Omega \times [0, T)$ with $\Omega \subseteq \mathscr{R}^3$ a bounded region with smooth boundary $\partial\Omega$ and $T > 0$ a finite real number. We assume also that the electric field and the electric displacement field satisfy initial data of the form

$$\mathbf{E}(\mathbf{x}, 0) = \mathbf{E}_0(\mathbf{x}), \qquad \mathbf{E}_t(\mathbf{x}, 0) = \mathbf{E}_1(\mathbf{x}) \tag{2.12}$$

$$\mathbf{D}(\mathbf{x}, 0) = \mathbf{D}_0(\mathbf{x}), \qquad \mathbf{D}_t(\mathbf{x}, 0) = \mathbf{D}_1(\mathbf{x}) \tag{2.13}$$

for all $x \in \Omega$, where $\mathbf{E}_0, \dots, \mathbf{D}_1$ are continuous functions on $\bar{\Omega}$, and homogeneous boundary data of the form

$$\mathbf{E}(x, t) = \mathbf{D}(x, t) = \mathbf{0}, \qquad (x, t) \in \partial\Omega \times [0, T). \tag{2.14}$$

In order to obtain the desired growth estimates for the systems consisting of (2.5), (2.12), (2.14a) and (2.6), (2.13), (2.14b), respectively, we first convert these initial-boundary value problems into initial value problems for abstract integrodifferential equations in an appropriate Hilbert space setting. Following Dafermos[1] [9] we denote by H, H_+ real Hilbert spaces with H_+ dense in H and $H_+ \subseteq H$ algebraically and topologically. The inner products on H, H_+ are denoted by $\langle\, ,\, \rangle$ and $\langle\, ,\, \rangle_+$, respectively. Let H_- denote the dual of H_+ via the inner product of H, i.e., H_- is the completion of H under the norm

$$\| \mathbf{w} \|_- = \sup_{\mathbf{v} \in H_+} \frac{|\langle \mathbf{v}, \mathbf{w} \rangle|}{\| \mathbf{v} \|_+} \tag{2.15}$$

and let $\mathscr{L}_s(H_+, H_-)$ be the space of symmetric bounded linear operators from H_+ into H_-. The abstract initial value problem we shall employ in this paper then has the form

$$\mathbf{u}_{tt} - \mathbf{N}\mathbf{u} + \int_{-\infty}^{t} \mathbf{K}(t - \tau)\, \mathbf{u}(\tau)\, d\tau = \mathbf{0}, \qquad 0 \leq t < T \tag{2.16}$$

$$\mathbf{u}(0) = \mathbf{f}, \qquad \mathbf{u}_t(0) = \mathbf{g} \tag{2.17}$$

$$\mathbf{u}(\tau) = \mathbf{0}, \qquad -\infty < \tau < 0 \tag{2.18}$$

where $\mathbf{u}:[0, T) \to H_+$, $\mathbf{f}, \mathbf{g} \in H_+$ and

(i) $\mathbf{N} \in \mathscr{L}_S(H_+, H_-)$

(ii) $\mathbf{K}(t), \mathbf{K}_t(t) \in L^2((-\infty, \infty); \mathscr{L}_S(H_+, H_-))$

with $\mathbf{K}_t$ denoting the strong operator derivative of $\mathbf{K}$. Now set

$$\mathscr{N} = \{\mathbf{w} \in C^2([0, T); H_+) \mid \sup_{[0,T)} \| \mathbf{w}(t) \|_+ \leq N^2\}, \tag{2.19}$$

[1] Sufficient conditions for the asymptotic stability of the fields $\mathbf{E}(t)$, $\mathbf{D}(t)$ may be deduced from Dafermos' work [9].

for some arbitrary real number N. Then the following specialization of a result due to Bloom [6] applies to any solution $\mathbf{u} \in C^2([0, T); H_+)$ of (2.16)–(2.18) for which $\mathbf{u}_t \in C^1([0, T); H_+)$ and $\mathbf{u}_{tt} \in C([0, T); H_-)$:

PROPOSITION I. *Let $\mathbf{u} \in \mathcal{N}$ be any solution of* (2.16)–(2.18) *and set*

$$F(t) = \| \mathbf{u}(t) \|^2 + \beta(t + t_0)^2, \qquad 0 \leqslant t < T \tag{2.20}$$

where β, t_0 are nonnegative real numbers. If $\mathbf{K}(t)$ satisfies

$$-\langle \mathbf{v}, \mathbf{K}(0)\, \mathbf{v} \rangle \geqslant \kappa \| \mathbf{v} \|_+^2 , \qquad \forall \mathbf{v} \in H_+ \tag{2.21}$$

with

$$\kappa \geqslant \gamma T \sup_{[0,T)} \| \mathbf{K}_t(t) \|_{\mathscr{L}(H_+, H_-)}{}^2 \tag{2.22}$$

then for all t, $0 \leqslant t < T$, $F(t)$ satisfies

$$F(t)\, F''(t) - [F'(t)]^2 \geqslant -2F(t)\, (2\mathscr{G}(0) + \beta) \tag{2.23}$$

where

$$\mathscr{G}(t) \equiv \mathscr{E}(t) + \theta \sup_{[0,T)} \| \mathbf{K}(t) \|_{\mathscr{L}(H_+, H_-)}{}^3 \tag{2.24}$$

and

$$\mathscr{E}(t) = \tfrac{1}{2}\langle \mathbf{u}_,(t), \mathbf{u}_t(t) \rangle - \tfrac{1}{2}\langle \mathbf{u}(t), \mathbf{N}\mathbf{u}(t) \rangle \tag{2.25}$$

Remarks. The proof of Proposition I, stated above, is given in [6] and proceeds via a logarithmic convexity argument due to Knops and Payne [10] for the special case in which $\mathbf{K}(t) \equiv 0$, $0 \leqslant t < T$. As no definiteness conditions are imposed on the operator $\mathbf{N}$ the technique is particularly well suited to handling certain non-well posed problems. We note in passing that the reader may easily check that the assumption of zero past history, i.e. (2.18), allows us to replace expressions such as $\sup_{[0,\infty)} \| \mathbf{K}(t) \|_{\mathscr{L}(H_+, H_-)}$ which appear in [6], [7], and [8] by supremums over the finite time interval $[0, T)$. As demonstrated in [6] and [7] all the growth estimates for the abstract system (2.16)–(2.18), which we shall employ in this paper, follow directly from the basic estimate (2.13).

We now recast our initial-boundary value problem (2.6), (2.13), (2.14b) for $\mathbf{D}(\mathbf{x}, t)$ into an initial-value problem of the form (2.16)–(2.18) as follows:[4]

Let $C_0^\infty(\Omega)$ denote the set of three dimensional vector fields with compact support in Ω whose components are in $C^\infty(\Omega)$. Following Dafermos [9] we

[2] γ is the embedding constant, i.e., as $H_+ \subset H$ topologically, $\| \mathbf{v} \| \leqslant \gamma \| \mathbf{v} \|_+$, $v \in H_+$, where, for the sake of convenience we will assume Ω to be such that $0 < \gamma < 1$; the reader can easy modify the ensuing analysis for the case where $\gamma > 0$.

[3] $\theta = \frac{1}{2}\gamma T N^4$.

[4] The argument follows the same pattern as that employed in [6] and [7] for the equations of motion for a three dimensional isothermal linear viscoelastic material.

define $\hat{H}$ to be the completion of $C_0^\infty(\Omega)$ under the norm induced by the inner product

$$\langle v, w\rangle_{\hat{H}} = \int_\Omega v_i w_i \, d\mathbf{x} \tag{2.26}$$

and take $\hat{H}_+$ to be completion of $C_0^\infty(\Omega)$ under the norm induced by the inner product

$$\langle \mathbf{v}, \mathbf{w}\rangle_{\hat{H}_+} = \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} \, d\mathbf{x} \tag{2.27}$$

Finally, $\hat{H}_-$ is defined to be the completion of $C_0^\infty(\Omega)$ under the norm

$$\| \mathbf{v} \|_{\hat{H}_-} = \sup_{\mathbf{w}\in H_+} [|\langle \mathbf{v}, \mathbf{w}\rangle_{\hat{H}}|/\| \mathbf{w} \|_{\hat{H}_+}]. \tag{2.28}$$

Operators $\hat{\mathbf{N}} \in \mathscr{L}_S(\hat{H}_+, \hat{H}_-)$ and $\hat{\mathbf{K}}(t) \in L^2((-\infty, \infty); \mathscr{L}_S(\hat{H}_+, \hat{H}_-))$ are now defined as follows

$$(\hat{\mathbf{N}}\mathbf{w})_i = \hat{N}_{ik} w_k \equiv \frac{1}{\epsilon\mu} \delta_{ik}\delta_{jl} \frac{\partial^2 w_k}{\partial x_j \, \partial x_l}, \qquad \forall \mathbf{w} \in \hat{H}_+ \tag{2.29}$$

and

$$(\hat{K}(t)\, \mathbf{w})_i \equiv -\Phi(t)\, \hat{N}_{ik} w_k, \qquad \forall \mathbf{w} \in \hat{H}_+ . \tag{2.30}$$

With these definitions of $\hat{H}$, $\hat{H}_+$, $\hat{H}_-$ and the operators $\hat{\mathbf{N}}$, $\hat{\mathbf{K}}$, the system consisting of (2.6), (2.13), and (2.14b) assumes the form

$$\mathbf{D}_{tt} - \hat{\mathbf{N}}\mathbf{D} + \int_{-\infty}^{t} \mathbf{K}(t-\tau)\, \mathbf{D}(\tau)\, d\tau = \mathbf{0} \tag{2.31}$$

$$\mathbf{D}(0) = \mathbf{D}_0, \qquad \mathbf{D}_t(0) = \mathbf{D}_1 \tag{2.32}$$

$$\mathbf{D}(\tau) = \mathbf{0}, \qquad -\infty < \tau < 0 \tag{2.33}$$

for $0 \leqslant t < T$, where $\mathbf{D}$: $[0, T) \to \hat{H}_+$, $\mathbf{D}_0$, $\mathbf{D}_1 \in \hat{H}_+$. We now seek to delineate the form which the conditions expressed by (2.21) and (2.22) assume in the present situation.

In terms of our definitions of $\hat{H}$ and $\hat{H}_+$, (2.21) assumes the form

$$-\int_\Omega v_i [\hat{\mathbf{K}}(0)\, \mathbf{v}]_i \, dx \geqslant \kappa \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\mathbf{x}, \qquad \forall \mathbf{v} \in \hat{H}_+ \tag{2.34}$$

or, in view of (2.30),

$$\frac{\Phi(0)}{\epsilon\mu} \int_\Omega \delta_{ik}\delta_{jl} v_i \frac{\partial^2 v_k}{\partial x_j \, \partial x_1} \, d\mathbf{x} \geqslant \kappa \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\mathbf{x}. \tag{2.35}$$

Integration of the expression on the left-hand side of (2.35) by parts, and an application of the divergence theorem, in conjunction with the fact that the vector field $\mathbf{v}$ vanishes[5] on $\partial\Omega$, $\forall v \in \hat{H}_+$, yields the result that (2.35) is equivalent to

$$\Phi(0) \leqslant -\kappa\epsilon\mu. \tag{2.36}$$

Note, however, that the hypotheses of Proposition I, i.e. (2.22) also require that

$$\kappa \geqslant \gamma T \cdot \sup_{[0,T)} \| \hat{\mathbf{K}}_t(t) \|_{\mathscr{L}(\hat{H}_+, \hat{H}_-)} \,. \tag{2.37}$$

A simple computation, however, yields

$$\begin{aligned}
|\langle \mathbf{v}, \hat{\mathbf{K}}_t(t)\, \mathbf{v} \rangle_{\hat{H}}| &= \left| \int_\Omega v_i [\hat{\mathbf{K}}_t(t)\, \mathbf{v}]_i \, d\mathbf{x} \right| \\
&= \frac{|\dot{\Phi}(t)|}{\epsilon\mu} \left| \int_\Omega \delta_{ij}\delta_{kl} v_i \frac{\partial^2 v_j}{\partial x_k \, \partial x_l} \, d\mathbf{x} \right| \\
&= \frac{|\dot{\Phi}(t)|}{\epsilon\mu} \left| \int_\Omega \frac{\partial v_i}{\partial x_j} \frac{\partial v_i}{\partial x_j} \, d\mathbf{x} \right| \equiv \frac{|\dot{\Phi}(t)|}{\epsilon\mu} \| \mathbf{v} \|^2_{\hat{H}_+}
\end{aligned} \tag{2.38}$$

for all $\mathbf{v} \in \hat{H}_+$, where we have again made use of the fact that $\mathbf{v}$ vanishes on $\partial\Omega$. However, by virtue of the Schwarz inequality and the definitions of $\hat{H}$, $\hat{H}_+$

$$\begin{aligned}
|\langle \mathbf{v}, \hat{\mathbf{K}}_t(t)\, \mathbf{v} \rangle_{\hat{H}}| &\leqslant \| \mathbf{v} \|_{\hat{H}} \| \hat{\mathbf{K}}_t(t)\, \mathbf{v} \|_{\hat{H}} \\
&\leqslant \| \hat{\mathbf{K}}_t(t) \|_{\mathscr{L}(\hat{H}_+, \hat{H}_-)} \| \mathbf{v} \|^2_{\hat{H}_+}
\end{aligned} \tag{2.39}$$

for all $\mathbf{v} \in \hat{H}_+$, since $\| \mathbf{v} \|_{\hat{H}} \leqslant \gamma \| \mathbf{v} \|_{\hat{H}_+}$, $\forall \mathbf{v} \in \hat{H}_+$, and we are assuming that $\gamma \in (0, 1)$. Therefore,

$$\| \hat{\mathbf{K}}_t(t) \|_{\mathscr{L}(\hat{H}_+, H_-)} = |\dot{\Phi}(t)| / \epsilon\mu, \qquad \forall t \in [0, T) \tag{2.40}$$

implying that (2.36) is to be restricted by the condition

$$\kappa \geqslant \frac{\gamma T}{\epsilon\mu} \sup_{[0,T)} |\dot{\Phi}(t)| \,. \tag{2.41}$$

Clearly (2.36), (2.41) are simultaneously satisfied if $\Phi(t)$, $0 \leqslant t < T$, satisfies

$$\Phi(0) \leqslant -\gamma T \sup_{[0,T)} |\dot{\Phi}(t)| \tag{2.42}$$

[5] This result follows via a standard trace theorem; I am indebted to Prof. S. Antman for this observation.

a condition to which we shall frequently return in the following sections. In view of Proposition I, and the discussion above, we have already established the following result:

THEOREM I. *Let*

$$\mathscr{M} = \left\{ \mathbf{d} \in C^2([0, T); \hat{H}_+) \mid \sup_{[0,T)} \left[\int_\Omega \frac{\partial d_i}{\partial x_j} \frac{\partial d_i}{\partial x_j} \, d\mathbf{x} \right]^{1/2} \leqslant M^2 \right\}$$

for some real number M, and let $\mathbf{D} \in \mathscr{M}$ be any solution of (2.6), (2.13), (2.14b). *If $\Phi(t)$ satisfies* (2.42) *then*

$$F(t; \beta, t_0) = \int_\Omega D_i(\mathbf{x}, t) \, D_i(\mathbf{x}, t) \, d\mathbf{x} + \beta(t + t_0)^2, \qquad 0 \leqslant t < T,$$

with β, t_0 nonnegative real numbers, satisfies

$$FF'' - F'^2 \geqslant -2F(2\hat{\mathscr{G}}(0) + \beta), \qquad 0 \leqslant t < T \tag{2.43}$$

where

$$\begin{aligned} \hat{\mathscr{G}}(t) = {} & \frac{1}{2} \int_\Omega \frac{\partial D_i(\mathbf{x}, t)}{\partial t} \frac{\partial D_i(\mathbf{x}, t)}{\partial t} \, d\mathbf{x} \\ & + \frac{1}{2\epsilon\mu} \int_\Omega \frac{\partial D_i(\mathbf{x}, t)}{\partial x_j} \frac{\partial D_i(\mathbf{x}, t)}{\partial x_j} \, d\mathbf{x} + \frac{\theta}{\epsilon\mu} \sup_{[0,T)} | \Phi(t) | . \end{aligned} \tag{2.44}$$

Remarks. Whereas we have written Theorem I out is some detail, we shall, for the most part, adhere to the Hilbert space notation in the remainder of the paper.

Remark. The most important thing to point out, at this stage of our analysis, is that although we may easily rewrite the integrodifferential equation (2.5) is the form,

$$\mathbf{E}_{tt} - \frac{1}{\epsilon} \left(\frac{1}{\mu} \Delta + \mathbf{I} \right) \mathbf{E} + \frac{1}{\epsilon} \int_0^t \phi_{tt}(t - \tau) \, \mathbf{E}(\tau) \, d\tau = - \frac{\phi(0)}{\epsilon} \mathbf{E}_t(t), \tag{2.45}$$

when $E(t) = 0$, $t < 0$, in order to be able to recast (2.45) in the Hilbert space setting already constructed we must have $\phi(0) = 0$; in this case we may rewrite the system consisting of (2.5), (2.12), and (2.14a) in the form

$$\mathbf{E}_{tt} - \mathbf{N}^*\mathbf{E} + \int_{-\infty}^t \mathbf{K}^*(t - \tau) \, \mathbf{E}(\tau) \, d\tau = \mathbf{0} \tag{2.46}$$

$$\mathbf{E}(0) = \mathbf{E}_0 , \qquad \mathbf{E}_t(0) = \mathbf{E}_1 \tag{2.47}$$

$$\mathbf{E}(\tau) = \mathbf{0}, \qquad -\infty < \tau < 0 \tag{2.48}$$

where $\mathbf{E}$: $[0, T) \to \hat{H}_+$, $\mathbf{E}_0$, $\mathbf{E}_1 \in \hat{H}_+$. The appropriate forms of the operators $\mathbf{N}^*$, $\mathbf{K}^*(t)$ appearing in (2.46) are

$$(\mathbf{N}^*\mathbf{w})_i = \frac{1}{\epsilon}\left(\frac{1}{\mu}\,\delta_{ik}\delta_{jl}\,\frac{\partial^2 w_k}{\partial x_j\,\partial x_l} + \delta_{ik}w_k\right) \tag{2.49}$$

$$(\mathbf{K}^*(t)\,\mathbf{w})_i = \frac{1}{\epsilon}\,\frac{d^2}{dt^2}\,\phi(t)\,\delta_{ij}w_j \tag{2.50}$$

where $\mathbf{w} \in \hat{H}_+$. It would than be possible to carry over most of the stability and growth estimates derived in [6] and [7] to the system (2.46)–(2.48). However, many experimental studies, including those of Hopkinson [3], indicate that suitable memory functions for various kinds of dielectrics, which are compatible with the basic constitutive equation (1.4), do not satisfy the condition that $\phi(0) = 0$. In particular, we have already mentioned Hopkinson's experimental attempts to verify a linear combination of exponential functions as being a reasonable memory function for glass and in this paper we shall be interested in applying some of our growth and stability estimates to the simple case where $\phi(t) = e^{-t}$. Our approach to the derivation of growth and stability estimates for the electric field $\mathbf{E}(\mathbf{x}, t)$ shall, therefore, be routed through the system (2.31)–(2.33). The results contained in [6] and [7] do yield growth and stability estimates for the electric displacement field $\mathbf{D}(\mathbf{x}, t)$; some corresponding theorems for the electric field may then be obtained by using the constitutive relations (1.4) and (2.1) and various estimates on the kernel functions which follow from (2.2)–(2.4) and which are derived in the next section.

3. Upper and Lower Bounds for $\sup_{[0,T)} |\Phi(t)|$ and $\sup_{[0,T)} |\dot{\Phi}(t)|$

Our first result in this section is the following

Lemma II. *Let $\phi(t) \in C^1[0, T)$ and assume that* (2.2), *and the series which is obtained from* (2.2) *by term by term differentiation, are both uniformly convergent,* $0 \leqslant t < T$. *Then provided* $\sup_{[0,T)} |\phi(t)| < \epsilon/T$ *we have*

$$\text{(a)}\quad \sup_{[0,T)} |\Phi(t)| \leqslant \alpha(T) \tag{3.1}$$

$$\text{(b)}\quad \sup_{[0,T)} |\dot{\Phi}(t)| \leqslant \frac{\alpha(T)}{T}\left(1 + T\,\frac{\sup_{[0,T)} |\dot{\phi}(t)|}{\sup_{[0,T)} |\phi(t)|}\right) \tag{3.2}$$

where

$$\alpha(T) \equiv \sup_{[0,T)} |\phi(t)|/(\epsilon - T \sup_{[0,T)} |\phi(t)|).$$

Proof. From (2.2)

$$|\Phi(t)| \leqslant \sum_{n=1}^{\infty} |\phi^n(t)|\,, \qquad 0 \leqslant t < T. \tag{3.3}$$

But from (2.3) and (2.4) we have, for $n \geqslant 2$,

$$\phi^n(t) = \frac{1}{\epsilon}\int_0^t \phi(t-\tau)\,\phi^{n-1}(\tau)\,d\tau, \qquad 0 \leqslant t < T \tag{3.4}$$

so

$$|\phi^n(t)| \leqslant \frac{T}{\epsilon} \sup_{(0,T]} |\phi(\tau)| \sup_{(0,T]} |\phi^{n-1}(\tau)| . \tag{3.5}$$

Since (3.5) is valid for all t, $0 \leqslant t < T$,

$$\sup_{[0,T)} |\phi^n(t)| \leqslant \frac{T}{\epsilon} \sup_{[0,T)} |\phi(t)| \sup_{[0,T)} |\phi^{n-1}(t)| \tag{3.6}$$

Successive application of the recursion formula (3.6) then yields

$$\begin{aligned}\sup_{[0,T)} |\phi^n(t)| &\leqslant \left(\frac{T}{\epsilon} \sup_{[0,T)} |\phi(t)|\right)^{n-1} \sup_{[0,T)} |\phi^1(t)| \\ &= \frac{1}{\epsilon}\left(\frac{T}{\epsilon} \sup_{[0,T)} |\phi(t)|\right)^{n-1} \sup_{[0,T)} |\phi(t)| \\ &= \frac{T^{n-1}}{\epsilon^n} (\sup_{[0,T)} |\phi(t)|)^n.\end{aligned} \tag{3.7}$$

Therefore, from (3.3) we have

$$|\Phi(t)| \leqslant \sum_{n=1}^{\infty} \frac{T^{n-1}}{\epsilon^n} (\sup_{[0,T)} |\phi|)^n, \qquad 0 \leqslant t < T \tag{3.8}$$

or

$$|\Phi(t)| \leqslant \frac{1}{T} \sum_{n=1}^{\infty} \left(\frac{T}{\epsilon} \sup_{[0,T)} |\phi(t)|\right)^n, \qquad 0 \leqslant t < T. \tag{3.9}$$

From our assumption that $\sup_{[0,T)} |\phi(t)| < \epsilon/T$ it follows that the geometric series on the right-hand side of (3.9) converges and, in fact, we have

$$|\Phi(t)| \leqslant \frac{1}{T}\left(\frac{T \sup_{[0,T)} |\phi(t)|}{\epsilon - T \sup_{[0,T)} |\phi(t)|}\right) \equiv \alpha(T) \tag{3.10}$$

for all t, $0 \leqslant t < T$, so that part (a) of Lemma II follows by taking the supremum over $[0, T)$ in (3.10). In order to prove part (b) of the lemma we beginning by noting that our hypotheses imply that

$$|\dot{\Phi}(t)| \leqslant \sum_{n=1}^{\infty} |\dot{\phi}^n(t)| , \qquad 0 \leqslant t < T. \tag{3.11}$$

However,

$$\begin{aligned}\dot{\phi}^n(t) &= \frac{1}{\epsilon}\, d/dt \int_0^t \phi(t-\tau)\,\phi^{n-1}(\tau)\,d\tau \\ &= \frac{\phi(0)}{\epsilon}\cdot\phi^{n-1}(t) + \frac{1}{\epsilon}\int_0^t \phi_t(t-\tau)\,\phi^{n-1}(\tau)\,d\tau\end{aligned} \tag{3.12}$$

for $0 \leqslant t < T$. Therefore,

$$\begin{aligned}|\dot{\phi}^n(t)| &\leqslant \frac{1}{\epsilon}\,|\phi(0)|\,|\phi^{n-1}(t)| + \frac{T}{\epsilon}\sup_{[0,T)}|\dot{\phi}(\tau)|\sup_{[0,T)}|\phi^{n-1}(\tau)| \\ &\leqslant \frac{1}{\epsilon}\left(|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|\right)\sup_{[0,T)}|\phi^{n-1}(\tau)|\end{aligned} \tag{3.13}$$

But, from the recursion formula (3.7_3)

$$\sup_{[0,T)}|\phi^{n-1}(\tau)| \leqslant \frac{T^{n-2}}{\epsilon^{n-1}}\left(\sup_{[0,T)}|\phi(\tau)|\right)^{n-1} \tag{3.14}$$

so that

$$|\dot{\phi}^n(t)| \leqslant \left(|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|\right)\frac{T^{n-2}}{\epsilon^n}\left(\sup_{[0,T)}|\phi(\tau)|\right)^{n-1}. \tag{3.15}$$

Substitution of (3.15) in (3.11) yields

$$|\dot{\Phi}(t)| \leqslant \left(|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|\right)\sum_{n=1}^{\infty}\frac{T^{n-2}}{\epsilon^n}\left(\sup_{[0,T)}|\phi(\tau)|\right)^{n-1} \tag{3.16}$$

for $0 \leqslant t < T$, or, replacing $|\phi(0)|$ by $\sup_{[0,T)}|\phi(t)|$

$$|\dot{\Phi}(t)| \leqslant \frac{\left(\sup_{[0,T)}|\phi(\tau)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|\right)}{T^2\sup_{[0,T)}|\phi(\tau)|}\cdot\sum_{n=1}^{\infty}\left(\frac{T\sup_{[0,T)}|\phi(\tau)|^n}{\epsilon}\right) \tag{3.17}$$

$$= \left(1 + T\frac{\sup_{[0,T)}|\dot{\phi}(\tau)|}{\sup_{[0,T)}|\phi(\tau)|}\right)\cdot\frac{1}{T^2}\,\frac{T\sup_{[0,T)}|\phi(\tau)|}{\left(\epsilon - T\sup_{[0,T)}|\phi(\tau)|\right)} \tag{3.18}$$

as we have assumed $\sup_{[0,T)}|\phi(\tau)| \in \epsilon/T$. Therefore,

$$|\dot{\Phi}(t)| \leqslant \frac{\alpha(T)}{T}\left(1 + T\frac{\sup_{[0,T)}|\dot{\phi}(\tau)|}{\sup_{[0,T)}|\phi(\tau)|}\right), \tag{3.19}$$

for $0 \leqslant t < T$, and the desired result follows by taking the supremum on the left-hand side of (3.19).

Remarks. We note here, in passing, an alternative method of deriving the results contained in Lemma II. We begin by multiplying (2.4) through by $(-1)^n$ and summing over n, $2 \leqslant n < \infty$, to get

$$\sum_{n=2}^{\infty} (-1)^n \phi^n(t) = \int_0^t \phi^1(t-\tau) \left[\sum_{n=0}^{\infty} (-1)^n \phi^{n-1}(\tau) \right] d\tau \tag{3.20}$$

where we have used our assumption of uniform convergence to interchange the integration and summation operations. But by (2.2) and (2.3)

$$\sum_{n=2}^{\infty} (-1)^n \phi^n(t) = \Phi(t) + \frac{1}{\epsilon} \Phi(t) \tag{3.21}$$

so (3.20) may be recast in the form

$$\begin{aligned} \Phi(t) + \frac{1}{\epsilon}\phi(t) &= -\frac{1}{\epsilon}\int_0^t \phi(t-\tau)\left[\sum_{n=2}^{\infty}(-1)^{n-1}\phi^{n-1}(\tau)\right] d\tau \\ &= -\frac{1}{\epsilon}\int_0^t \phi(t-\tau)\,\Phi(\tau)\,d\tau. \end{aligned} \tag{3.22}$$

As a direct consequence of (3.22_2) we have $\Phi(0) = -(1/\epsilon)\,\phi(0)$. If we differentiate (3.22_2) through with respect to t now we obtain

$$\epsilon\dot{\Phi}(t) = -\dot{\phi}(t) - \phi(0)\,\Phi(t) - \int_0^t \phi_t(t-\tau)\,\Phi(\tau)\,d\tau, \tag{3.23}$$

a result which will be employed in the proof of Lemma III. Note that (3.23), in conjunction with $\Phi(0) = -(1/\epsilon)\,\phi(0)$, implies that $\Phi(0) = -(1/\epsilon)\,\dot{\phi}(0) + (1/\epsilon^2)\,\phi^2(0)$.

In order to establish part (a) of Lemma II, we rewrite (3.22_2) in the form

$$\Phi(t) = -\frac{1}{\epsilon}\phi(t) - \frac{1}{\epsilon}\int_0^t \phi(t-\tau)\,\Phi(\tau)\,d\tau. \tag{3.24}$$

Then

$$\begin{aligned} |\Phi(t)| &\leqslant \frac{1}{\epsilon}\,|\phi(t)| + \frac{1}{\epsilon}\sup_{[0,T)}|\phi(\tau)| \int_0^t |\Phi(\tau)|\,d\tau \\ &\leqslant \frac{1}{\epsilon}\sup_{[0,T)}|\phi(\tau)|\,(1 + T\sup_{[0,T)}|\Phi(\tau)|) \end{aligned} \tag{3.25}$$

or, since (3.25) is valid for all t, $0 \leqslant t < T$,

$$\epsilon \sup_{[0,T)} |\Phi(t)| \leqslant \sup_{[0,T)} |\phi(t)|\,(1 + T\sup_{[0,T)}|\Phi(t)|) \tag{3.26}$$

which, in turn, may be rewritten as

$$\sup_{[0,T)} |\Phi(t)| \, (\epsilon - T \sup_{[0,T)} |\phi(t)|) \leqslant \sup_{[0,T)} |\phi(t)| \,, \qquad 0 \leqslant t < T. \tag{3.27}$$

Finally, as we are assuming that $\sup_{[0,T)} |\phi(t)| < \epsilon/T$, we may divide both sides of (3.27) through by $\epsilon - T \sup_{[0,T)} |\phi(t)|$ to obtain the desired result; the result contained in part (b) of Lemma II may be obtained from (3.23) in an analogous manner.

Our next lemma gives lower bounds for $\sup_{[0,T)} |\Phi(t)|$ and $\sup_{[0,T)} |\dot{\Phi}(t)|$ in terms of $\sup_{[0,T)} |\phi(t)|$ and $\sup_{[0,T)} {}_0\dot{\phi}(t)|$.

Lemma III. *Under the conditions which prevail in Lemma* II

(a) $$\sup_{[0,T)} |\Phi(t)| \geqslant \sup_{[0,T)} |\phi(t)|/\chi_T \tag{3.28}$$

where $\chi_T = \epsilon + T \sup_{[0,T)} |\phi(t)|$. *If, additionally,*

$$\sup_{[0,T)} |\dot{\phi}(t)| \geqslant |\phi(0)|^2/(\epsilon - T |\phi(0)|) \tag{3.29}$$

then

(b) $$\sup_{[0,T)} |\dot{\Phi}(t)| \geqslant \frac{\sup_{[0,T)} |\dot{\phi}(t)| \, (\epsilon - T |\phi(0)|) - |\phi(0)|^2}{2\epsilon^2 + \epsilon T^2 \sup_{[0,T)} |\dot{\phi}(t)|} \,.$$

Proof. In order to prove part (a) recall that by virtue of (3.22_2)

$$\phi(t) = -\epsilon\Phi(t) - \int_0^t \phi(t-\tau)\, \Phi(\tau)\, d\tau, \qquad 0 \leqslant t < T \tag{3.30}$$

so that

$$\begin{aligned} |\phi(t)| &\leqslant \epsilon |\Phi(t)| + \sup_{[0,T)} |\phi(\tau)| \int_0^t |\Phi(\lambda)|\, d\lambda \\ &\leqslant \epsilon |\Phi(t)| + T \sup_{[0,T)} |\phi(\tau)| \sup_{[0,T)} |\Phi(\tau)| \\ &\leqslant (\epsilon + T \sup_{[0,T)} |\phi(\tau)|) \sup_{[0,T)} |\Phi(\tau)| \\ &= \chi_T \sup_{[0,T)} |\Phi(\tau)| \,. \end{aligned} \tag{3.31}$$

Therefore, taking the supremum over $[0, T)$ in (3.31_4) we get

$$\sup_{[0,\tau)} |\phi(t)| \leqslant \chi_T \sup_{[0,T)} |\Phi(t)| \tag{3.32}$$

and as $\chi_T > 0$ the desired result follows immediately. We now assume that in addition to the other hypotheses of Lemma II, the estimate (3.29) also holds. If we solve (3.23) for $\dot{\phi}$ we obviously get

$$\dot{\phi}(t) = -\epsilon\dot{\Phi}(t) - \phi(0)\,\Phi(t) - \int_0^t \phi_t(t-\tau)\,\Phi(\tau)\,d\tau \tag{3.33}$$

for all t, $0 \leqslant t < T$. Thus,

$$|\dot{\phi}(t)| \leqslant \epsilon\,|\dot{\Phi}(t)| + (|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|)\sup_{[0,T)}|\Phi(\tau)|\,. \tag{3.34}$$

However, $\Phi(t) = \int_0^t \dot{\Phi}(\tau)\,d\tau + \Phi(0)$, so

$$|\Phi(t)| \leqslant T\sup_{[0,T)}|\dot{\Phi}(\tau)| + \frac{1}{\epsilon}|\phi(0)| \tag{3.35}$$

where we have made use of the relation between $\phi(0)$ and $\Phi(0)$. Since (3.35_2) holds for $0 \leqslant t < T$, we have

$$\sup_{[0,T)}|\Phi(\tau)| < T\sup_{[0,T)}|\dot{\Phi}(\tau)| + \frac{1}{\epsilon}|\phi(0)| \tag{3.36}$$

and substitution of this result into (3.4) yields

$$\begin{aligned}|\dot{\phi}(t)| &\leqslant \epsilon\,|\dot{\Phi}(t)| + (|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|)\left(T\sup_{[0,T)}|\dot{\Phi}(\tau)| + \frac{1}{\epsilon}|\phi(0)|\right)\\ &\leqslant \sup_{[0,T)}|\dot{\Phi}(\tau)|\,(\epsilon + T[|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|])\\ &\quad + \frac{1}{\epsilon}|\phi(0)|\,(|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|)\end{aligned} \tag{3.37}$$

Taking the supremum over $[0, T)$ on the left-hand side of (3.37_2), and rearranging terms, we get

$$\frac{\sup_{[0,T)}|\dot{\phi}(\tau)|\left(1 - \frac{T}{\epsilon}|\phi(0)|\right) - \frac{1}{\epsilon}|\phi(0)|^2}{\epsilon + T(|\phi(0)| + T\sup_{[0,T)}|\dot{\phi}(\tau)|)} \leqslant \sup_{[0,T)}|\dot{\Phi}(t)|\,. \tag{3.38}$$

Note that our requirement that $\sup_{[0,T)}|\phi(\tau)| < \epsilon/T$ implies that the coefficient of $\sup_{[0,T)}|\dot{\phi}(t)|$, in the numerator of the expression on the left-hand side of (3.38), is positive (as is the numerator itself in view of (3.29)). Therefore

$$\begin{aligned}\sup_{[0,T)}|\dot{\Phi}(t)| &\geqslant \frac{\sup_{[0,T)}|\dot{\phi}(t)|\,(\epsilon - T|\phi(0)|) - |\phi(0)|^2}{\epsilon^2 + \epsilon T|\phi(0)| + \epsilon T^2\sup_{[0,T)}|\dot{\phi}(\tau)|}\\ &\geqslant \frac{\sup_{[0,T)}|\dot{\phi}(t)|\,(\epsilon - T|\phi(0)|) - |\phi(0)|^2}{2\epsilon^2 + \epsilon T^2\sup_{[0,T)}|\dot{\phi}(\tau)|}\end{aligned} \tag{3.39}$$

where we have used the fact that $T\,|\,\phi(0)| < \epsilon$. This establishes part (b) of the lemma.

EXAMPLE. In order to examine the implications of Lemmas II and III we consider the simple example $\phi(t) = e^{-t}$, $0 \leqslant t < T$, and denote the corresponding Φ as $\Phi(\tau; e^{-t})$. Since $\sup_{[0,T)} |\,\phi(t)| = 1$, the condition that $\sup_{[0,T)} |\,\phi(t)| < \epsilon/T$ is equivalent to the condition that $T < \epsilon$; if this simple inequality is satisfied then part (a) of Lemma II implies that

$$\sup_{0\leqslant\tau<T} |\,\Phi(\tau; e^{-t})| \leqslant \frac{1}{\epsilon - T}; \qquad T < \epsilon. \tag{3.40}$$

Clearly, $\sup_{[0,T)} |\,\dot{\phi}(t)| = 1$, so part (b) of Lemma II yields

$$\sup_{0\leqslant\tau<T} |\,\dot{\Phi}(\tau; e^{-t})| \leqslant \frac{1+T}{T(\epsilon - T)}; \qquad T < \epsilon. \tag{3.41}$$

Turning now to Lemma III we again require that $T < \epsilon$; part (a) then yields the lower bound

$$\sup_{0\leqslant\tau<T} |\,\Phi(\tau; e^{-t})| \geqslant \frac{1}{\epsilon + T}; \qquad T < \epsilon \tag{3.42}$$

while for part (b) of Lemma III we must require that (3.29) be satisfied, i.e., that $\epsilon - T > 1$. We thus have the lower bound

$$\sup_{0\leqslant\tau<T} |\,\dot{\Phi}(\tau; e^{-t})| \geqslant \frac{\epsilon - T - 1}{2\epsilon^2 + \epsilon T^2}; \qquad 1 < \epsilon - T. \tag{3.43}$$

Clearly (3.43) requires that $\epsilon > 1$; in addition, there is obviously no need to also specify that $T < \epsilon$ since this is automatically satisfied whenever the condition implying the validity of the estimate in (3.43) is.

Remark. In the example considered above, i.e., $\phi(t) = e^{-t}$, the condition expressed by (2.42) becomes

$$-\frac{1}{\epsilon}\phi(0) = -\frac{1}{\epsilon} \leqslant -\gamma T \sup_{0\leqslant\tau<T} |\,\dot{\Phi}(\tau, e^{-t})|\,. \tag{3.44}$$

However, from (3.41) we have $\sup_{0\leqslant\tau<T} |\,\dot{\Phi}(\tau; e^{-t})| \leqslant (1 + T)/T(\epsilon - T)$, if $T < \epsilon$. Thus (3.44) will be satisfied if

$$\frac{1}{\epsilon} \geqslant \gamma T \cdot \frac{(1+T)}{T(\epsilon - T)} = \frac{\gamma(1+T)}{\epsilon - T} \tag{3.45}$$

and simple manipulation shows that (3.45) is equivalent to

$$T \leqslant \frac{\epsilon(1-\gamma)}{\epsilon\gamma + 1} \Rightarrow T \leqslant \epsilon(\gamma \in (0, 1)). \tag{3.46}$$

4. Growth Theorems for Electric Fields in Nonconducting Material Dielectrics

Our first result, is based upon the following specialization of a theorem obtained in [6] for the abstract system (2.16)–(2.18):

Proposition II. *Let* $\mathbf{u} \in \mathcal{N}$ *be any solution of* (2.16)–(2.18) *for which* $\mathscr{E}(0) \leqslant -k$ *for some* $k > 0$. *If* $\mathbf{K}(t)$ *satisfies* (2.21), (2.22) *and*

$$\sup_{[0,T)} \| \mathbf{K}(t)\|_{\mathscr{L}(H_+,H_-)} \leqslant k/\theta \tag{4.1}$$

then, provided $\langle \mathbf{f}, \mathbf{g} \rangle > 0$,

$$\| \mathbf{u}(t)\|^2 \geqslant \| \mathbf{f} \|^2 \exp\{\langle 2\mathbf{f}, \mathbf{g}\rangle\, t/\| \mathbf{f} \|^2\}, \qquad 0 \leqslant t < T. \tag{4.2}$$

In view of the identification which we have already made between the abstract system (2.16)–(2.18) and the initial-boundary value problem (2.6), (2.13), (2.14b), we can immediately state

Theorem II. *Let* $\mathbf{D} \in \mathscr{M}$ *be any solution of* (2.6), (2.13), (2.14b) *with the class* $\mathscr{M}$ *as defined in Theorem* I, *Section* 2, *and suppose that* $\Phi(t)$ *satisfies* (2.42). *If*

$$\| \mathbf{D}_1 \|_{\hat{H}}^2 - \langle \mathbf{D}_0 , \hat{\mathbf{N}}\mathbf{D}_0\rangle_{\hat{H}} \leqslant -2k \tag{4.3a}$$

for some $k > 0$ *and*

$$\sup_{[0,T)} | \Phi(t)| \leqslant \epsilon\mu k/\theta \tag{4.3b}$$

then, provided $\langle \mathbf{D}_0 , \mathbf{D}_1 \rangle_{\hat{H}} > 0$,

$$\| \mathbf{D}(t)\|_{\hat{H}}^2 \geqslant \| \mathbf{D}_0 \|_{\hat{H}}^2 \exp\{\langle 2\mathbf{D}_0 , D_1\rangle_{\hat{H}}\, t/\| \mathbf{D}_0 \|_{\hat{H}}^2\}, \qquad 0 \leqslant t < T. \tag{4.4}$$

In order to obtain the corresponding growth theorem for solutions of the initial boundary value problem (2.5), (2.12), (2.14a) we proceed as follows: from the constitutive equation of Hopkinson, i.e. (1.4), and the assumption that $\mathbf{E}(\tau) = 0$, $-\infty < \tau < 0$, we have for $0 \leqslant t < T$

$$\begin{aligned}
\| \mathbf{D}(t)\|_{\hat{H}} &\leqslant \epsilon \| \mathbf{E}(t)\|_{\hat{H}} + \int_0^t | \phi(t - \tau)| \, \| \mathbf{E}(t)\|_{\hat{H}}\, d\tau \\
&\leqslant \epsilon \| \mathbf{E}(t)\|_{\hat{H}} + \sup_{[0,T)} | \phi(\tau)| \int_0^t \| \mathbf{E}(\tau)\|_{\hat{H}}\, d\tau \\
&\leqslant \chi_T \sup_{[0,T)} \| \mathbf{E}(\tau)\|_{\hat{H}} .
\end{aligned} \tag{4.5}$$

Now, directly from (2.1) we have

$$\mathbf{D}_0 = \epsilon \mathbf{E}_0 \tag{4.6}$$

and

$$\dot{\mathbf{E}}(t) = \frac{1}{\epsilon}\dot{\mathbf{D}}(t) + \frac{1}{\epsilon}\left[\int_0^t \Phi_t(t-\tau)\,\mathbf{D}(\tau)\,d\tau + \Phi(0)\,\mathbf{D}(t)\right] \tag{4.7}$$

from which (as $\Phi(0) = -(1/\epsilon)\,\phi(0)$) we easily obtain

$$\mathbf{D}_1 = \epsilon\left(\mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right). \tag{4.8}$$

Therefore, condition (4.3a) is equivalent to

$$\left\|\mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}}\mathbf{E}_0\rangle_{\hat{H}} \leqslant -2k/\epsilon^2, \qquad k > 0. \tag{4.9}$$

On the other hand, (2.42) is equivalent to

$$\phi(0) \geqslant \epsilon\gamma T \sup_{[0,T)} |\dot{\Phi}(t)| \tag{4.10}$$

in view of the stated relation between the initial values of ϕ and Φ. Suppose now that $\phi(t) \in C^1[0, T)$, than $\sup_{[0,T)} |\phi(t)| < \epsilon/T$ and that both (2.2) and the derived series, which is obtained from (2.2) via term by term differentiation, are uniformly convergent, $0 \leqslant t < T$. Then, as a direct consequence of Lemma IIb, (4.10) will be satisfied if

$$\phi(0) \geqslant \epsilon\gamma\alpha(T)\left(1 + T\,\frac{\sup_{[0,T)} |\dot{\phi}(t)|}{\sup_{[0,T)} |\phi(t)|}\right). \tag{4.11}$$

Under the same conditions stated above, it follows from Lemma IIa that (4.3b) will be satisfied if

$$\alpha(T) \leqslant \epsilon\mu k/\theta. \tag{4.12}$$

In view of Theorem II, and the above discussion, we may state our first growth estimate for the electric field, viz.,

COROLLARY I. *Let* $\mathbf{E} \in \mathscr{M}$ *be any solution of* (2.5), (2.12), (2.14a), *and suppose that the hypotheses of Lemma* II *are satisfied. If*

(i) *the initial data* $\mathbf{E}_0$, $\mathbf{E}_1$ *satisfy* (4.9) *for some* $k > 0$,

(ii) $\phi(t)$ *satisfies* (4.11) *and* (4.12)

(iii) $\phi(0)/\epsilon \,\|\mathbf{E}_0\|_{\hat{H}}^2 + \langle \mathbf{E}_0, \mathbf{E}_1\rangle_{\hat{H}} > 0$

then for all t, $0 \leqslant t < T$,

$$\sup_{[0,T)} \|\mathbf{E}(\tau)\|_{\hat{H}} \geqslant \frac{\epsilon}{\chi_T}\|\mathbf{E}_0\|_{\hat{H}} \exp\left\{\frac{\langle \mathbf{E}_0, \mathbf{E}_1 + [\phi(0)/\epsilon]\,\mathbf{E}_0\rangle_{\hat{H}}\, t}{\|\mathbf{E}_0\|_{\hat{H}}^2}\right\}. \tag{4.16}$$

EXAMPLE. Consider the simple case $\phi(t) = e^{-t}$. We have already seen in Section 3 that $\alpha(T) = 1/(\epsilon - T)$, $\sup_{[0,T)} |\phi(\tau)| < \epsilon/T$ if and only if $T < \epsilon$, and that (2.42) is satisfied if $T \leqslant \epsilon(1-\gamma)/(\epsilon\gamma + 1)$; this latter condition is of course, equivalent to (4.11) in this case. If we use the definition of θ, it is a simple matter to show that (4.12) is satisfied if and only if $T \leqslant \epsilon^2\psi/(1 + \epsilon\psi)$, where $\psi = 2\mu k/\gamma N^4$. We may, therefore, specialize Corollary I, as follows: Let $\mathbf{E} \in \mathscr{M}$ be any solution of (2.5), (2.12) (2.14a) with $\phi(t) = e^{-t}$. If

$$\text{(i}'\text{)} \quad \left\| \mathbf{E}_1 + \frac{1}{\epsilon}\mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}}\mathbf{E}_0 \rangle_{\hat{H}} \leqslant -2k/\epsilon^2, \qquad k \geqslant 0 \tag{4.14}$$

$$\text{(ii}'\text{)} \quad T \leqslant \min\left\{ \frac{\epsilon(1-\gamma)}{\epsilon\gamma + 1}, \frac{\epsilon^2\psi}{1 + \epsilon\psi} \right\}, \qquad \psi = 2\mu k/\gamma N^4 \tag{4.15}$$

$$\text{(iii}'\text{)} \quad \| \mathbf{E}_0 \|_{\hat{H}}^2 > -\epsilon\langle \mathbf{E}_0, \mathbf{E}_1 \rangle_{\hat{H}} \tag{4.16}$$

then for all t, $0 \leqslant t < T$,

$$\sup_{[0,T)} \| \mathbf{E}(\tau) \|_{\hat{H}} \geqslant \left(\frac{\epsilon}{\epsilon + T} \right) \| \mathbf{E}_0 \|_{\hat{H}} \exp\left\{ \frac{\langle \mathbf{E}_0, \mathbf{E}_1 + (1/\epsilon)\mathbf{E}_0 \rangle_{\hat{H}}\, t}{\| \mathbf{E}_0 \|_{\hat{H}}^2} \right\}. \tag{4.17}$$

Our next set of growth estimates is based upon the following specialization of a theorem derived in [6]:

PROPOSITION III. *Let* $\mathbf{u} \in \mathscr{N}$ *be any solution of* (2.16)–(2.18) *for which* $\mathscr{E}(0) \geqslant -\tilde{k}$, *for some* $\tilde{k} > 0$ *and suppose that the initial data satisfy*

$$\langle \mathbf{f}, \mathbf{g} \rangle \geqslant (2\mathscr{G}(0))^{1/2} \| \mathbf{f} \|. \tag{4.18}$$

If $\mathbf{K}(t)$ *satisfies* (2.21), (2.22) *and, in addition,*

$$\sup_{[0,T)} \| \mathbf{K}_t(t) \|_{\mathscr{L}(H_+, H_-)} > \tilde{k}/\theta T, \tag{4.19}$$

then for all t, $0 \leqslant t < T$

$$\| \mathbf{u}(t) \|^2 \geqslant \left[\| \mathbf{f} \|^2 + \frac{4\mathscr{G}(0)}{\lambda^2} \right] \cosh \lambda t + \left[\frac{\langle 2\mathbf{f}, \mathbf{g} \rangle}{\lambda} \right] \sinh \lambda t - \frac{4\mathscr{F}(0)}{\lambda^2} \tag{4.20}$$

provided

$$\lambda^2 = \left(\frac{\langle 2\mathbf{f}, \mathbf{g} \rangle}{\| \mathbf{f} \|^2} \right)^2 - \frac{8\mathscr{G}(0)}{\| \mathbf{f} \|^2} \neq 0. \tag{4.21}$$

If $\lambda^2 = 0$, *then under the conditions stated above*

$$\| \mathbf{u}(t) \|^2 \geqslant \| \mathbf{f} \|^2 + 2(2\mathscr{G}(0))^{1/2} \| \mathbf{f} \|\, t + 2\mathscr{G}(0)\, t^2, \qquad 0 \leqslant t < T. \tag{4.22}$$

In view of the identification which has been established between the system (2.16)–(2.18) and the initial-boundary value problem (2.6), (2.13), and (2.14b) we have

THEOREM III. *Let* $\mathbf{D} \in \mathscr{M}$ *be any solution of* (2.6), (2.13), (2.14b) *and suppose that* $\Phi(t)$ *satisfies* (2.42). *Suppose, also, that the initial data* $\mathbf{D}_0$, $\mathbf{D}_1$ *satisfy*

$$\| \mathbf{D}_1 \|_{\hat{H}}^2 - \langle \mathbf{D}_0, \hat{\mathbf{N}}\mathbf{D}_0 \rangle_{\hat{H}} \geqslant -2\tilde{k}, \tag{4.23}$$

for some $\tilde{k} > 0$ *and*[6]

$$\langle \mathbf{D}_0, \mathbf{D}_1 \rangle_{\hat{H}} \geqslant (2\hat{\mathscr{G}}(0))^{1/2} \| \mathbf{D}_0 \|_{\hat{H}}, \tag{4.24a}$$

$$\hat{\mathscr{G}}(0) = \frac{1}{2} (\| \mathbf{D}_1 \|_{\hat{H}}^1 - \langle \mathbf{D}_0, \hat{\mathbf{N}}\mathbf{D}_0 \rangle_{\hat{H}}) + \frac{\theta}{\epsilon\mu} \sup_{[0,T)} | \Phi(t)| . \tag{4.24b}$$

If

$$\sup_{[0,T)} | \dot{\Phi}(t)| > \tilde{k}\epsilon\mu/\theta T \tag{4.25}$$

then for all t, $0 \leqslant t < T$,

$$\| \mathbf{D}(t)\|_{\hat{H}}^2 \geqslant \left[\| \mathbf{D}_0 \|_{\hat{H}}^2 + \frac{4\hat{\mathscr{G}}(0)}{\hat{\lambda}^2}\right] \cosh \hat{\lambda}t + \left(\frac{\langle 2\mathbf{D}_0, \mathbf{D}_1 \rangle_{\hat{H}}}{\hat{\lambda}}\right) \sinh \hat{\lambda}t - \frac{4\hat{\mathscr{G}}(0)}{\hat{\lambda}^2} \tag{4.26}$$

when

$$\hat{\lambda}^2 \equiv \left(\frac{\langle 2\mathbf{D}_0, \mathbf{D}_1 \rangle_{\hat{H}}}{\| \mathbf{D}_0 \|_{\hat{H}}^2}\right)^2 - \frac{8\hat{\mathscr{G}}(0)}{\| \mathbf{D}_0 \|_{\hat{H}}^2} \neq 0.$$

If $\hat{\lambda}^2 = 0$, *then*

$$\| \mathbf{D}(t)\|_{\hat{H}}^2 \geqslant \| \mathbf{D}_0 \|_{\hat{H}}^2 + 2(2\hat{\mathscr{G}}(0))^{1/2} \| \mathbf{D}_0 \|_{\hat{H}}\, t + 2\hat{\mathscr{G}}(0)\, t^2, \qquad 0 \leqslant t < T. \tag{4.27}$$

To obtain from this last theorem two new growth estimates for the electric field, we will employ the estimate (4.5_3) and will also assume that the conditions Lemma II are satisfied. In addition, as a consequence of Lemma III(b), (4.25) is implied by

$$\frac{\sup_{[0,T)} | \dot{\phi}(t)| \,(\epsilon - T\, | \phi(0)|) - | \phi(0)|^2}{2\epsilon^2 + \epsilon T^2 \sup_{[0,T)} | \phi(t)|} \geqslant \frac{\tilde{k}\epsilon\mu}{\theta T} \tag{4.28a}$$

$$\sup_{[0,T)} | \dot{\phi}(t)| \geqslant \frac{| \phi(0)|^2}{\epsilon - T\, | \phi(0)|} . \tag{4.28b}$$

[6] We remark that (2.42) and (4.25) guarantee that $\hat{\mathscr{G}}(0) > 0$ so that (4.24a) makes sense.

Thus there remains only the problem of restating (4.23) and (4.24) in terms of the initial data $\mathbf{E}_0$, $\mathbf{E}_1$ associated with the electric field and the function $\phi(t)$. To this end we note that in view of (4.6), (4.8), and Lemma IIa

$$\begin{aligned}\hat{\mathscr{G}}(0) &= \frac{\epsilon^2}{2}\left(\left\|\mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}}\mathbf{E}_0\rangle_{\hat{H}}\right) + \frac{\theta}{\epsilon\mu}\sup_{[0,T)} |\Phi(t)| \\ &= \frac{\epsilon^2}{2}\left(\left\|\mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}}\mathbf{E}_0\rangle_{\hat{H}}\right) + \frac{\theta\alpha(T)}{\epsilon\mu} \equiv \hat{\mathscr{G}}^*.\end{aligned} \tag{4.29}$$

For future reference we also note the estimate

$$\hat{\mathscr{G}}(0) \geqslant \frac{\epsilon^2}{2}\left(\left\|\mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}}\mathbf{E}_0\rangle_{\hat{H}}\right) + \frac{\theta}{\epsilon\mu\chi_T}\cdot\sup_{[0,T)} |\phi(t)| \equiv \hat{\mathscr{G}}_* \tag{.430}$$

which follows directly from Lemma IIIa. In view of (4.29) it is clear that (4.24a) is implied by

$$\left\langle \mathbf{E}_0, \mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right\rangle_{\hat{H}} \geqslant \frac{(2\hat{\mathscr{G}}^*)^{1/2}}{\epsilon}\|\mathbf{E}_0\|_{\hat{H}} \tag{4.31}$$

and it is easy to check that strict inequality in (4.31) implies that $\hat{\lambda}^2 > 0$. We summarize our results in

Corollary II. *Let* $\mathbf{E} \in \mathscr{M}$ *be any solution of* (2.5), (2.12), (2.14a) *and suppose that the hypotheses of Lemma* II *are satisfied. If the initial data* $\mathbf{E}_0$, $\mathbf{E}_1$ *satisfy* (4.33), *with* $\hat{\mathscr{G}}^*$ *as given in* (4.29), *and*

$$\left\|\mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}}\mathbf{E}_0\rangle_{\hat{H}} \geqslant -2\tilde{k}/\epsilon^2, \tag{4.32}$$

for some $\tilde{k} > 0$, *while* $\phi(t)$ *satisfies* (4.11), (4.28a), *and* (4.28b), *then for all* t, $0 \leqslant t < T$

$$\begin{aligned}\left(\sup_{[0,T)} \|\mathbf{E}(\tau)\|_{\hat{H}}\right)^2 \geqslant \frac{1}{{\chi_T}^2}\Bigg\{&\left[\epsilon^2\|\mathbf{E}_0\|_{\hat{H}}^2 + \frac{4\hat{\mathscr{G}}(0)}{\hat{\lambda}^2}\right]\cosh\hat{\lambda}t \\ &+ \left[\frac{2\epsilon^2}{\hat{\lambda}^2}\left\langle \mathbf{E}_0, \mathbf{E}_1 + \frac{\phi(0)}{\epsilon}\mathbf{E}_0\right\rangle_{\hat{H}}\right]\sinh\hat{\lambda}t - \frac{4\hat{\mathscr{G}}(0)}{\hat{\lambda}^2}\Bigg\}\end{aligned} \tag{4.33}$$

where

$$\hat{\lambda}^2 = 4\left(\frac{\langle \mathbf{E}_0, \mathbf{E}_1 + [\phi(0)/\epsilon]\mathbf{E}_0\rangle_{\hat{H}}}{\|\mathbf{E}_0\|_{\hat{H}}^2}\right)^2 - \frac{8\hat{\mathscr{G}}(0)}{\epsilon^2\|\mathbf{E}_0\|_{\hat{H}}^2}. \tag{4.34}$$

If we allow for the possibility that $\phi(t)$ and the initial data may satisfy $\hat{\lambda}^2 = 0$, with $\hat{\lambda}$ defined by (4.34), then we have

COROLLARY III. *Let $E \in \mathcal{M}$ be any solution of* (2.5), (2.12), *and* (2.14a). *Suppose that the hypotheses of Lemma II are satisfied, that the initial data satisfy* (4.32), *for some $\tilde{k} > 0$, and that*

$$\left\langle \mathbf{E}_0, \mathbf{E}_1 + \frac{\phi(0)}{\epsilon} \mathbf{E}_0 \right\rangle_{\hat{H}} = \frac{(2\hat{\mathscr{G}}(0))^{1/2}}{\epsilon} \| \mathbf{E}_0 \|_{\hat{H}} . \tag{4.35}$$

If $\phi(t)$ satisfies (4.11), (4.28a), *and* (4.28b), *the for all t, $0 \leqslant t < T$*

$$(\sup_{[0,T)} \| \mathbf{E}(\tau) \|_{\hat{H}})^2 \geqslant \frac{1}{\chi_T{}^2} \{\epsilon^2 \| \mathbf{E}_0 \|_{\hat{H}}^2 + (2\epsilon(2\hat{\mathscr{G}}(0))^{1/2}) \| \mathbf{E}_0 \|_{\hat{H}}\, t + 2\hat{\mathscr{G}}(0)\, t^2\}. \tag{4.36}$$

EXAMPLE. We return to our example, i.e., $\phi(t) = e^{-t}$. As we have already seen, $\alpha(T) = 1/(\epsilon - T)$, $\chi_T = \epsilon + T$, and $\sup_{[0,T)} |\phi(t)| < \epsilon/T$ if and only if $T < \epsilon$; the other conditions of Lemma II are, of course, clearly satisfied. We easily compute that

$$\hat{\mathscr{G}}^* = \hat{\mathscr{g}}^* \equiv \frac{\epsilon^2}{2} \left\{ \left\| \mathbf{E}_1 + \frac{1}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}} \mathbf{E}_0 \rangle_{\hat{H}} \right\} + \frac{\theta}{\mu\epsilon(\epsilon - T)} \tag{4.37a}$$

$$\hat{\mathscr{G}}_* = \hat{\mathscr{g}}_* \equiv \frac{\epsilon^2}{2} \left\{ \left\| \mathbf{E}_1 + \frac{1}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0, \hat{\mathbf{N}} \mathbf{E}_0 \rangle_{\hat{H}} \right\} + \frac{\theta}{\mu\epsilon(\epsilon + T)} . \tag{4.37b}$$

Condition (4.31) then assumes the form

$$\left\langle \mathbf{E}_0, \mathbf{E}_1 + \frac{1}{\epsilon} \mathbf{E}_0 \right\rangle_{\hat{H}} > \frac{(2\hat{\mathscr{g}}^*)^{1/2}}{\epsilon} \| \mathbf{E}_0 \|_{\hat{H}} . \tag{4.38}$$

However, the conclusion of Corollary II is valid if (4.31) is replaced by the weaker condition

$$\left\langle \mathbf{E}_0, \mathbf{E}_1 + \frac{1}{\epsilon} \mathbf{E}_0 \right\rangle_{\hat{H}} \geqslant \frac{(2\hat{\mathscr{G}}(0))^{1/2}}{\epsilon} \| \mathbf{E}_0 \|_{\hat{H}} \tag{4.39}$$

(this obviously being sufficient to guarantee that $\hat{\lambda}^2 \geqslant 0$) and as we may proceed directly with the computation of $\hat{\mathscr{G}}(0)$ in this example, we shall have no further recourse to (4.38). Before embarking upon the computation of $\hat{\mathscr{G}}(0)$ let us recall that $\phi(t) = e^{-t}$ will satisfy (4.11) if $T \leqslant \epsilon(1 - \gamma)/(\epsilon\gamma + 1)$; also (4.28a) and (4.28b) reduce, in this case to the simple inequalities

$$\frac{T(\epsilon - T - 1)}{\epsilon^2(2\epsilon + T^2)} > \frac{\tilde{k}\mu}{\theta} \qquad \text{and} \qquad \epsilon > T + 1 \tag{4.40}$$

so that our results are valid only for $\epsilon > 1$. We now compute $\sup_{0 \leqslant \tau < T} | \Phi(\tau; e^-{}_\nu)|$ under the assumption that $\epsilon > 1$. Directly from the definition of $\phi^n(\tau)$, $n \geqslant 2$, i.e., (2.4), and the fact that $\phi^1(\tau) = (1/\epsilon)\, e^{-\tau}$

$$\phi^n(\tau) = \frac{1}{\epsilon^{n-1}}\, e^{-\tau} \frac{\tau^{n-1}}{(n-1)!}, \qquad n \geqslant 2. \tag{4.41}$$

Therefore,

$$\begin{aligned}
\Phi(\tau; e^{-t}) &= \sum_{n-1}^{\infty} (-1)^n \phi^n(\tau) \\
&= -\frac{1}{\epsilon} e^{-\tau} + e^{-\tau} \sum_{n=2}^{\infty} (-1)^n \frac{1}{e^{n-1}} \frac{\tau^{n-1}}{(n-1)!} \\
&= -\frac{1}{\epsilon} e^{-\tau} + e^{-\tau} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\tau^n}{\epsilon^n n!} \qquad (4.42) \\
&= -e^{-\tau} \left(\frac{1}{\epsilon} + \sum_{n=1}^{\infty} \frac{(-\tau/\epsilon)^n}{n!} \right) \\
&= -e^{-\tau} \left(\frac{1}{\epsilon} + [e^{-\tau/\epsilon} - 1] \right)
\end{aligned}$$

where $0 \leqslant \tau < T < \epsilon$ so that $| \tau/\epsilon | < 1$. Clearly, we may put this last result in the form

$$\Phi(\tau; e^{-t}) = (1 - 1/\epsilon)\, e^{-\tau} - e^{-\alpha\tau}; \qquad \alpha = \frac{1 + \epsilon}{\epsilon} \tag{4.43}$$

and the following facts are then easy to verify:

$$\text{(i)} \quad \Phi(0, e^{-t}) = -1/\epsilon < 0; \qquad \Phi'(0; e^{-t}) = 2/\epsilon > 0 \tag{4.44}$$

$$\text{(ii)} \quad \Phi(\tau; e^{-t}) > 0 \Leftrightarrow e^{\tau/\epsilon} > \epsilon/(\epsilon - 1) \tag{4.45}$$

$$\text{(iii)} \quad \Phi'(\tau; e^{-t}) > 0 \Leftrightarrow \frac{\epsilon + 1}{\epsilon - 1} > e^{\tau/\epsilon}. \tag{4.46}$$

In other words, $\Phi(\tau; e^{-t})$ is initially negative and increasing and continues to increase on the interval $(0, \epsilon \ln[(\epsilon + 1)/(\epsilon - 1)])$, becoming positive for $\tau > \epsilon \ln[\epsilon/(\epsilon - 1)]$.

Therefore,

$$\sup_{0 \leqslant \tau < T} | \Phi(\tau; e^{-t})| = \max\{1/\epsilon; \lim_{\tau \to T^-} | \Phi(\tau; e^{-t})|\} \tag{4.47}$$

provided we choose

$$T < \epsilon \ln \left[\frac{\epsilon + 1}{\epsilon - 1} \right]. \tag{4.48}$$

Set $\pi(\epsilon) = \epsilon \ln[(\epsilon + 1)/(\epsilon - 1)]$. A simple computation then shows that $\lim_{\tau \to T^-} | \Phi(\tau; e^{-t})| < 1/\epsilon$. In fact if $\lim_{\tau \to T^-} \Phi(\tau; e^{-t}) < 0$, then by virtue of of (4.48) and the monotonicity of Φ on $[0, \pi(\epsilon))$

$$0 > \lim_{\tau \to T^-} \Phi(\tau; e^{-t}) > \Phi(0; e^{-t}) = -1/\epsilon \tag{4.49}$$

from which it is clear that $\lim_{\tau \to T^-} | \Phi(\tau; e^{-t})| < 1/\epsilon$.

If $\lim_{\tau\to T^-} \Phi(\tau; e^{-t}) > 0$ then again by virtue of (4.48) and the monotonicity of Φ

$$0 < \lim_{\tau\to T^-} \Phi(\tau; e^{-t}) < \Phi(\pi(\epsilon); e^{-t}). \tag{4 50}$$

But

$$\begin{aligned}\Phi(\pi(\epsilon); e^{-t}) &= (1 - 1/\epsilon)\, e^{-\pi(\epsilon)} - e^{-\alpha\pi(\epsilon)} \\ &= (1 - 1/\epsilon) \exp\left(\ln\left[\frac{\epsilon+1}{\epsilon-1}\right]^{-\epsilon}\right) - \exp\left([1 + 1/\epsilon] \ln\left[\frac{\epsilon+1}{\epsilon-1}\right]^{-\epsilon}\right)\end{aligned} \tag{4.51}$$

or

$$\begin{aligned}\Phi(\pi(\epsilon); e^{-t}) &= \left(1 - \frac{1}{\epsilon}\right)\left(\frac{\epsilon+1}{\epsilon-1}\right)^{-\epsilon} - \left(\frac{\epsilon+1}{\epsilon-1}\right)^{-(\epsilon+1)} \\ &= \left(\frac{\epsilon+1}{\epsilon-1}\right)^{-\epsilon}\left[\left(1 - \frac{1}{\epsilon}\right) - \left(\frac{\epsilon+1}{\epsilon-1}\right)^{-1}\right] \\ &\left(\frac{\epsilon-1}{\epsilon-1}\right)^{-\epsilon} \cdot \frac{\epsilon-1}{\epsilon(\epsilon+1)} \\ &= \frac{1}{\epsilon}\left(\frac{\epsilon-1}{\epsilon+1}\right)^{\epsilon+1} < 1/\epsilon.\end{aligned} \tag{4.52}$$

as $((\epsilon - 1)/(\epsilon - 1)) < 1$ and $\epsilon > 1$. It thus follows directly from (4.47) that

$$\sup_{0\leqslant\tau<T} |\Phi(\tau; e^{-t})| = \frac{1}{\epsilon}. \tag{4.53}$$

Therefore, directly from (4.29) we obtain the value

$$\hat{\mathscr{G}}(0) = \hat{g} \equiv \frac{\epsilon^2}{2}\left(\left\|\mathbf{E}_1 + \frac{1}{\epsilon}\mathbf{E}_0\right\|_{\hat{H}}^2 - \langle\mathbf{E}_0, \hat{\mathbf{N}}\mathbf{E}_0\rangle_{\hat{H}}\right) + \frac{\theta}{\epsilon^2\mu}, \tag{4.54}$$

which is valid when T is restricted by (4.48); the corresponding value of $\hat{\lambda}^2$ is now determined via substitution of (4.54) in (4.34), after setting $\phi(0) = 1$. We note in passing, that (4.48) implies that $T < \epsilon$ if $\epsilon > 1 + 2/(e - 1)$.

We may sum up the preceding discussion, relative to the example $\phi(t) = e^{-t}$, as follows: Let $\mathbf{E} \in \mathscr{M}$ be any solution of (2.5), (2.12), (214a) with $\phi(t) = e^{-t}$ and $\epsilon > 1$ and choose T so that

$$T < \min\left\{\frac{\epsilon(1-\gamma)}{\epsilon\gamma+1}, \epsilon \ln\left[\frac{\epsilon+1}{\epsilon-1}\right]\right\}. \tag{4.55}$$

Suppose that the initial data $\mathbf{E}_0$, $\mathbf{E}_1$ satisfy

$$\left\|\mathbf{E}_1 + \frac{1}{\epsilon}\mathbf{E}_0\right\|_{\hat{H}}^2 - \langle\mathbf{E}_0, \mathbf{N}\mathbf{E}_0\rangle_{\hat{H}} \geqslant -2\tilde{k}/\epsilon^2, \qquad \tilde{k} > 0 \tag{4.56}$$

$$\left\langle\mathbf{E}_0, \mathbf{E}_1 + \frac{1}{\epsilon}\mathbf{E}_0\right\rangle_{\hat{H}} > \frac{(2\hat{g})^{1/2}}{\epsilon} \|\mathbf{E}_0\|_{\hat{H}} \tag{4.57}$$

with $\hat{g}$ defined by (4.54). Then if (4.40) is satisfied

$$(\sup_{[0,T)} \| \mathbf{E}(\tau) \|_{\hat{H}})^2$$
$$\geqslant \frac{4\hat{g}}{\hat{\lambda}^2(\epsilon + T)^2} (\cosh \hat{\lambda} t - 1)$$
$$+ \frac{1}{(\epsilon + T)^2} \left\{ \epsilon^2 \| \mathbf{E}_0 \|_{\hat{H}}^2 \cosh \hat{\lambda} t + \frac{2\epsilon^2}{\hat{\lambda}^2} \left\langle \mathbf{E}_0 , \mathbf{E}_1 + \frac{1}{\epsilon} \mathbf{E}_0 \right\rangle_{\hat{H}} \sinh \hat{\lambda} t \right\} \tag{4.58}$$

for all t, $0 \leqslant t < T$, where $\hat{\lambda}$ is given by (4.34) with $\hat{\mathscr{G}}(0) = \hat{g}$ and $\phi(0) = 1$. On the other hand, if instead of (4.57), the initial data satisfy

$$\left\langle \mathbf{E}_0 , \mathbf{E}_1 + \frac{1}{\epsilon} \mathbf{E}_0 \right\rangle_{\hat{H}} = \frac{(2\hat{g})^{1/2}}{\epsilon} \| \mathbf{E}_0 \|_{\hat{H}} \tag{4.59}$$

then for all t, $0 \leqslant t < T$

$$(\sup_{[0,T)} \| \mathbf{E}(\tau) \|_{\hat{H}})^2 \geqslant \frac{1}{(\epsilon + T)^2} \{ \epsilon^1 \| \mathbf{E}_0 \|_{\hat{H}}^2 + (2\epsilon(2\hat{g})^{1/2}) \| \mathbf{E}_0 \|_{\hat{H}} \, t + 2\hat{g} t^2 \}. \tag{4.60}$$

5. Stability Estimates for Electric Fields in Non-Conducting Material Dielectrics

In this section we derive some stability estimates and upper bounds to complement the growth theorems and lower bounds of the previous section; our main result is based upon the following specialization, to the abstract system (2.16)–(2.18), of a stability estimate derived in [7] and subsequently applied to study the stability of an isothermal linear viscoelastic body:

Proposition IV. *Let* $\mathbf{u} \in \mathscr{N}$ *be any solution of* (2.16)–(2.18) *and assume that* $\mathbf{K}(t)$ *satisfies* (2.21), (2.22). *If*

$$\dot{\mathbf{E}}(0) \leqslant -\hat{k} \sup_{[0,T)} \| \mathbf{K}(t) \|_{\mathscr{L}(H_+, H_-)}, \qquad \hat{k} \geqslant \theta \tag{5.1}$$

then for all t, $0 \leqslant t < T$

$$\| \mathbf{u}(t) \|^2 \leqslant A[\max(\| \mathbf{f} \|^2, \| \mathbf{g} \|^2)]^{2(1-\delta)} \tag{5.2}$$

where $A > 0$ *is independent of* t *and* $\delta = t/T$.

As with our two previous propositions, it can be shown [7] that this last result is a direct consequence of Proposition I. Furthermore, in view of the established analogy between the initial-boundary value which governs the evolution of the electric displacement field $\mathbf{D}(t)$ and the abstract system (2.16)–(2.18), we can state

THEOREM IV. *Let $\mathbf{D} \in \mathcal{M}$ be any solution of* (2.6), (2.13), (2.14b) *and suppose that $\Phi(t)$ satisfies* (2.42). *If the initial data $\mathbf{D}_0$, $\mathbf{D}_1$ satisfy*

$$\|\mathbf{D}_1\|_{\hat{H}}^2 - \langle \mathbf{D}_0, \hat{\mathbf{N}}\mathbf{D}_0 \rangle_{\hat{H}} \leqslant \frac{-2\hat{k}}{\epsilon\mu} \sup_{[0,T)} |\Phi(t)|, \tag{5.3}$$

for some $\hat{k} \geqslant 0$, then for all t, $0 \leqslant t < T$

$$\|\mathbf{D}(t)\|_{\hat{H}}^2 \leqslant \hat{A}[\max(\|\mathbf{D}_0\|_{\hat{H}}^2, \|\mathbf{D}_1\|_{\hat{H}}^2)]^{2(1-\delta)} \tag{5.4}$$

where $\hat{A} > 0$ is independent of t and $\delta = t/T$.

In order to obtain the implications of this last theorem for the behavior of the electric field $\mathbf{E}(t)$ we now make use of (2.1). In fact, directly from (2.1) we obtain the simple estimates

$$\begin{aligned}
\|\mathbf{E}(t)\|_{\hat{H}} &\leqslant \frac{1}{\epsilon} \|\mathbf{D}(t)\|_{\hat{H}} + \frac{1}{\epsilon} \int_0^t |\Phi(t-\tau)| \, \|\mathbf{D}(\tau)\|_{\hat{H}} \, d\tau \\
&\leqslant \frac{1}{\epsilon} \|\mathbf{D}(t)\|_{\hat{H}} + \frac{t}{\epsilon} \sup_{[0,T)} |\Phi(\tau)| \sup_{[0,T)} \|\mathbf{D}(\tau)\|_{\hat{H}} \\
&\leqslant \frac{1}{\epsilon} (1 + T \sup_{[0,T)} |\Phi(\tau)|) \sup_{[0,T)} \|\mathbf{D}(\tau)\|_{\hat{H}} .
\end{aligned} \tag{5.5}$$

However, provided that the conditions of Theorem IV are satisfied

$$\sup_{[0,t)} \|\mathbf{D}(\tau)\|_{\hat{H}} \leqslant \hat{A}^{1/2} \sup_{[0,t)} (\max[\|\mathbf{D}_0\|_{\hat{H}}^2, \|\mathbf{D}_1\|_{\hat{H}}^2])^{1-\tau/T} \tag{5.6}$$

for all t, $0 \leqslant t < T$. Recalling now the relations (4.6) and (4.8), and assuming the validity of Lemma II, we may combine (5.5) and (5.6) to obtain the estimates

$$\begin{aligned}
&\|\mathbf{E}(t)\|_{\hat{H}} \\
&\leqslant \frac{\hat{A}^{1/2}}{\epsilon} (1 + T \sup_{[0,T)} |\Phi(\tau)|) \sup_{[0,T)} \left(\epsilon^2 \max \left[\|\mathbf{E}_0\|_{\hat{H}}^2, \left\|\mathbf{E}_1 + \frac{\phi(0)}{\epsilon} \mathbf{E}_0\right\|_{\hat{H}}^2\right]\right)^{1-\tau/T} \\
&\leqslant \frac{\hat{A}^{1/2}}{\epsilon} (1 + T\alpha(T)) \sup_{[0,T)} \left(\epsilon^2 \max \left[\|\mathbf{E}_0\|_{\hat{H}}^2, \left\|\mathbf{E}_1 + \frac{\phi(0)}{\epsilon} \mathbf{E}_0\right\|_{\hat{H}}^2\right]\right)^{1-\tau/T}
\end{aligned} \tag{5.7}$$

for all t, $0 \leqslant t < T$. There therefore remains for us the simple task of determining conditions on $\mathbf{E}_0$, $\mathbf{E}_1$, and $\phi(t)$ which will ensure the validity of the various hypotheses of Theorem IV. We have already established that, granted the validity of Lemma II, the fact that $\phi(t)$ satisfies (4.11) implies that $\Phi(t)$ satisfies (2.42). Also, the condition on the initial data, which is expressed by (5.3), is easily seen to assume the form

$$\left\| \mathbf{E}_1 + \frac{}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0 , \hat{\mathbf{N}}\mathbf{E}_0 \rangle_{\hat{H}} \leqslant \frac{-2\hat{k}}{\epsilon^3 \mu} \sup_{[0,T)} | \Phi(t) | , \qquad \hat{k} \geqslant \theta. \tag{5.8}$$

But, in view of Lemma II(a), (5.5) is satisfied if

$$\left\| \mathbf{E}_1 + \frac{\phi(0)}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0 , \hat{\mathbf{N}}\mathbf{E}_0 \rangle_{\hat{H}} \leqslant \frac{-2\hat{k}\alpha(T)}{\epsilon^3 \mu} , \qquad \hat{k} \geqslant \theta. \tag{5.9}$$

We summarize the preceding discussion as

COROLLARY IV. *Let* $\mathbf{E} \in \mathscr{M}$ *be any solution of* (2.5), (2.12), (2.14a) *and suppose that the hypotheses of Lemma* II *are satisfied. If* $\phi(t)$ *satisfies* (4.11) *and the initial data* $\mathbf{E}_0$, $\mathbf{E}_1$ *satisfy* (5.9), *for some* $\hat{k} \geqslant \theta$, *then for all* t, $0 \leqslant t < T$, $\mathbf{E}(t)$ *satisfies*

$$\| \mathbf{E}(t) \|_{\hat{H}} \leqslant \frac{\beta(T)}{\epsilon} \sup_{[0,T)} \left(\epsilon^2 \max \left[\| \mathbf{E}_0 \|_{\hat{H}}^2 , \left\| \mathbf{E}_1 + \frac{\phi(0)}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 \right] \right)^{1-\tau/T} \tag{5.10}$$

where

$$\beta(T) = \hat{A}^{1/2} \left(1 + \frac{T \sup_{[0,T)} | \phi(t) |}{\epsilon - T \sup_{[0,T)} | \phi(t) |} \right)$$

and $\hat{A} > 0$ *is independent of* t.

EXAMPLE. We return again to the simple example $\phi(t) = e^{-t}$, for which the hypotheses of Lemma II are satisfied if $T < \epsilon$. In fact, we have already determined that with $T < \epsilon$

$$\Phi(\tau; e^{-t}) = \left(1 - \frac{1}{\epsilon} \right) e^{-\tau} - e^{-\lambda\tau}; \qquad \alpha = \frac{1 + \epsilon}{\epsilon}.$$

Furthermore, $\phi(t)$ satisfies (4.11) if we choose $T \leqslant \epsilon(1 - \gamma)/(\epsilon\gamma + 1)$. As $\alpha(T) = 1/(\epsilon - T)$ the restriction (5.9), on the initial data, assumes the form

$$\left\| \mathbf{E}_1 + \frac{1}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0 , \hat{\mathbf{N}}\mathbf{E}_0 \rangle_{\hat{H}} \leqslant \frac{-2\hat{k}}{\mu\epsilon^3(\epsilon - T)} \tag{5.11}$$

with $\hat{k} \geqslant \theta$. A particularly simple and elegant stability estimate now appears in the special case where $\mathbf{E}_1 = \mathbf{0}$: Let $\mathbf{E} \in \mathscr{M}$ be any solution of (2.5), (2.12), (2.14a) with $\phi(t) = e^{-t}$ and $T \leqslant \epsilon(1 - \gamma)/(\epsilon\gamma + 1)$. If $\mathbf{E}_1 = \mathbf{0}$ and $\mathbf{E}_0$ satisfies

$$\| \mathbf{E}_0 \|_{\hat{H}}^2 - \epsilon^2 \langle \mathbf{E}_0 , \hat{\mathbf{N}}\mathbf{E}_0 \rangle_{\hat{H}} \leqslant -2\hat{k}/\mu\epsilon(\epsilon - T), \qquad \hat{k} \geqslant \theta \tag{5.12}$$

there exists $\hat{A} > 0$ (independent of t) such that for all $t \in [0, T)$

$$\| \mathbf{E}(t) \|_{\hat{H}} \leqslant \frac{\hat{A}^{1/2}}{(\epsilon - T)} \sup_{[0,t)} (\epsilon \| \mathbf{E}_0 \|_{\hat{H}})^{2(1-\tau/T)} \qquad (\epsilon > 1) \tag{5.13}$$

$$\| \mathbf{E}(t) \|_{\hat{H}} \leqslant \frac{\hat{A}^{1/2}}{(\epsilon - T)} \sup_{[0,t)} \| \mathbf{E}_0 \|_{\hat{H}}^{2(1-\tau/T)} \qquad (\epsilon \leqslant 1). \tag{5.14}$$

Our other stability estimate is also a consequence of a specialization, to the abstract system (2.16)–(2.18), of a result derived in [7], namely

Proposition V. *Let $\mathbf{u} \in \mathscr{N}$ be any solution of* (2.16)–(2.18). *Assume that* $\mathbf{K}(t)$ *satisfies* (2.21), (2.22) *and that* $\mathscr{E}(0) \geqslant -\bar{k}$ *for some* $\bar{k} \geqslant 0$. *If*

$$\text{(i)} \quad \sup_{[0,T)} \| \mathbf{K}_t(t) \|_{\mathscr{L}(H_+, H_-)} \geqslant \bar{k}/\theta\gamma T \tag{5.15a}$$

and

$$\text{(ii)} \quad \lim_{T\to\infty} \frac{1}{T} \ln\{\| \mathbf{u}(T) \|^2 + \beta(T + t_0)^2\} = 0, \tag{5.15b}$$

for $\boldsymbol{\beta}$, t_0 *nonnegative constants satisfying* $\beta t_0{}^2 \leqslant \| \mathbf{f} \|^2$, *then*

$$\| \mathbf{u}(t) \|^2 \leqslant \Psi(t_0; T; \delta) \| \mathbf{f} \|^2, \qquad 0 \leqslant t < T \tag{5.16}$$

where

$$\Psi(t_0; T; \tau) \doteqdot 2(T/t_0 + 1)^{2+\delta} \qquad \textit{with} \qquad \delta = \frac{\mathscr{G}(0)}{\beta}.$$

From Proposition V we immediately deduce

Theorem V. *Let $\mathbf{D} \in \mathscr{M}$ be any solution of* (2.6), (2.13), (2.14b). *Assume that* $\Phi(t)$ *satisfies* (2.42) *and that*

$$\| \mathbf{D}_1 \|_{\hat{H}}^2 - \langle \mathbf{D}_0 , \hat{\mathbf{N}} \mathbf{D}_0 \rangle_{\hat{H}} \geqslant -2\bar{k} \tag{5.17}$$

for some $\bar{k} \geqslant 0$. *If*

$$\text{(i)} \quad \sup_{[0,T)} | \dot{\Phi}(t) | \geqslant \epsilon\mu\bar{k}/\theta\gamma T \tag{5.18}$$

$$\text{(ii)} \quad \lim_{T\to\infty} \frac{1}{T} \ln\{\| \mathbf{D}(T) \|_{\hat{H}}^2 + \beta(T + t_0)^2\} = 0, \tag{5.19}$$

for β, t_0 *nonnegative constants satisfying* $\beta t_0{}^2 \leqslant \| \mathbf{D}_0 \|_{\hat{H}}^2$, *then*

$$\| \mathbf{D}(t) \|_{\hat{H}}^2 \leqslant \hat{\Psi}(t_0; T; \hat{\delta}) \| \mathbf{D}_0 \|_{\hat{H}}^2, \qquad 0 \leqslant t < T \tag{5.20}$$

where $\hat{\Psi}(t_0; T; \hat{\delta}) = 2(T/t_0 + 1)^{2+\hat{\delta}}$ *with*

$$\hat{\delta} = \frac{\hat{\mathscr{G}}(0)}{\beta} = \frac{1}{2\beta} \left(\| \mathbf{D}_1 \|_{\hat{H}}^2 - \langle \mathbf{D}_0 , \hat{\mathbf{N}} \mathbf{D}_0 \rangle_{\hat{H}} + \frac{2\theta}{\epsilon\mu} \sup_{[0,T)} | \Phi(\tau) | \right).$$

In order to deduce some implications of Theorem V for the growth behavior of the electric field, we proceed as follows: First of all, (2.42) is again implied by (4.11) (if the conditions guaranteeing the validity of Lemma II are satisfied) and the restriction (5.17), on the initial data, clearly assumes the equivalent form

$$\left\| \mathbf{E}_1 + \frac{\phi(0)}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0 , \hat{\mathbf{N}}\mathbf{E}_0 \rangle_{\hat{H}} \geqslant -2\bar{k}/\epsilon^2, \tag{5.21}$$

with $\bar{k} \geqslant 0$. If, in addition to the hypotheses of Lemma II, we know that $\phi(t)$ satisfies (3.29), then it follows from Lemma III(b) that (5.18) is implied by the inequality

$$\frac{\sup_{[0,T)} |\phi(t)| (\epsilon - T |\phi(0)|) - |\phi(0)|^2}{\epsilon^2(2\epsilon + T^2 \sup_{[0,T)} |\dot{\phi}(t)|)} \geqslant \frac{\mu\bar{k}}{\theta\gamma T}. \tag{5.22}$$

As the natural logarithm is monotonically increasing and (by virtue of (4.8))

$$\| \mathbf{D}(T) \|_{\hat{H}} = \lim_{\tau\to T^-} \| \mathbf{D}(\tau) \|_{\hat{H}} \leqslant \sup_{[0,T)} \| \mathbf{D}(\tau) \|_{\hat{H}} \leqslant \chi_T \sup_{[0,T)} \| \mathbf{E}(\tau) \|_{\hat{H}}, \tag{5.23}$$

it is clear that (5.19) will be satisfied if

$$\lim_{T\to\infty} \frac{1}{T} \ln\{\chi_T{}^2(\sup_{[0,T)} \| \mathbf{E}(\tau) \|_{\hat{H}})^2 + \beta(T + t_0)^2\} = 0 \tag{5.24}$$

with β, t_0 nonnegative constants satisfying

$$\beta(t_0/\epsilon)^2 \leqslant \| \mathbf{E}_0 \|_{\hat{H}}^2 . \tag{5.25}$$

Finally, we note that, in view of Lemma II(a)

$$\delta \leqslant \delta^* = \frac{\epsilon^2}{2\beta} \left(\left\| \mathbf{E}_1 + \frac{\phi(0)}{\epsilon} \mathbf{E}_0 \right\|_{\hat{H}}^2 - \langle \mathbf{E}_0 , \mathbf{N}\mathbf{E}_0 \rangle_{\hat{H}} + \frac{2\theta\alpha(T)}{\epsilon^3\mu} \right). \tag{5.26}$$

Combining our results with the estimates (5.20) and (5.5_3) yields

COROLLARY V. *Let $\mathbf{E} \in \mathcal{M}$ be any solution of* (2.5), (2.12), (2.14a). *Suppose that the hypotheses of Lemma* II *are satisfied and that Φ satisfies* (2.42) *and* (5.18). *If the initial data satisfy* (5.21), *with $\bar{k} \geqslant 0$, and* (5.24), *for β, t_0 nonnegative constants satisfying* (5.25), *then*

$$\| \mathbf{E}(t) \|_{\hat{H}} \leqslant (1 + T\alpha(T)) (\Psi^*)^{1/2} \| \mathbf{E}_0 \|_{\hat{H}}, \qquad 0 \leqslant t < T \tag{5.27}$$

with $\Psi^*(t_0; T, \delta^*) = 2(T/t_0 + 1)^{2+\delta^*}$ and δ^* given by (5.26).

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